Illustration - 44 *Let X is a set containing n elements. A subset P of set X is chosen at random. The set X is then reconstructed by replacing the elements of set P and another set Q is chosen at random. Find number of ways to select P and Q such that $P \cap Q$ is empty i.e. $P \cap Q = \phi$.*

(A) 3^n (B) 2^n (C) $2^n - 1$ (D) $3^n - 1$

SOLUTION : (A)

$P \cap Q = \phi$. It means P and Q should be disjoint sets. That is there is no element common in P and Q.

$\Rightarrow$ For every elements in set X there are 3 choices. Either it is selected in P but not in Q or selected in Q but not in P or not selected in both P and Q.

$\Rightarrow$ Number of ways to select P and Q such that $P \cap Q$ is $\phi = 3^n$

Illustration - 45 *Let X is a set containing n elements. A subset P of set X is chosen at random. The set X is then reconstructed by replacing the elements of set P and another set Q is chosen at random. Find number of ways to select P and Q such that $P = \overline{Q}$.*

(A) 3^n (B) 2^n (C) $2^n - 1$ (D) $3^n - 1$

SOLUTION : (B)

$P = \overline{Q}$. It means P and Q are complementary sets i.e. every element present in X is either present in P or Q

$\Rightarrow$ For every element there are 2 choices to select. Either it will be selected for P or it will be selected for Q.

$\Rightarrow$ No. of ways to select $= 2^n$

DIVISION AND DISTRIBUTION OF NON-IDENTICAL ITEMS — Section - 6

6.1 Case - I : Unequal division and distribution of non-identical objects

In this section, we will discuss ways to divide non-identical objects into groups. For example, if we have to divide three different balls (b_1, b_2, b_3) among 2 boys (B_1 and B_2) such that B_1 gets 2 balls and B_2 gets 1 ball, then

Number of ways to divide balls among boys is 3 ways as shown in the following table.

B_1	B_2
b_1, b_2	b_3
b_2, b_3	b_1
b_3, b_1	b_2

Instead of writing all ways and counting them, we can make a formula to find number of ways.

First select 2 balls for B_1 in ${}^3C_2$ and then remaining 1 ball for B_2 in ${}^1C_1$ ways.

Total number of ways, using fundamental principle of counting, is

$$= {}^3C_2 \times {}^1C_1 = 3 \times 1 = 3 \text{ ways.}$$

If we have to divide 3 non-identical balls among 2 boys such that one boy should get 2 and other boy should get 1, then following are the ways :

B_1	B_2
b_1, b_2	b_3
b_2, b_3	b_1
b_3, b_1	b_2
b_3	b_1, b_2
b_1	b_2, b_3
b_2	b_3, b_1

Distribution of above 3 ways among 2 boys. You can observe that entries are interchanged, between B_1 and B_2.

$\Rightarrow$ Total ways to distribute = 6.

Instead of writing all ways and counting them, we can just find number of ways using fundamental principle of counting.

First select 2 balls for B_1 in ${}^3C_2$ ways, then select 1 remaining ball for B_2 in ${}^1C_1$ ways, finally distribute the ways to divide among 2 boys in $\lfloor 2$ ways (ball given to B_1 and B_2 are interchanged) because any boy can get 2 balls and the other 1 ball.

Using fundamental principle of counting, total number of ways

$$= {}^3C_2 \times {}^1C_1 \times \lfloor 2 = 3 \times 1 \times 2 = 6 \text{ ways.}$$

Now generalising the above cases, we can write the following formula :

(a) Number of ways in which $(m + n + p)$ different objects can be divided into 3 unequal groups (groups contain unequal number of objects) containing m, n, p objects $= {}^{m+n+p}C_m \, {}^{n+p}C_n \, {}^pC_p = \dfrac{(m+n+p)!}{m!\,n!\,p!}$

(b) Number of ways in which $(m + n + p)$ different objects can be divided and distributed (entries are distributed among groups) into 3 unequal groups (groups contain unequal number of objects) containing m, n, p objects

= No. of ways to divide $(m + n + p)$ objects in 3 groups × No. of ways to distribute 'division-ways' among groups

= No. of ways to divide $(m + n + p)$ objects in 3 groups × (Number of groups) ! $= \dfrac{(m+n+p)!}{m!\,n!\,p!} \times 3!$

Above formulae are written for dividing objects into 3 groups but in case groups are more, then also we follow the same approach. For example,

Number of ways to divide non-identical objects in 4 groups (G_1, G_2, G_3, G_4) such that groups G_1, G_2, G_3, G_4 gets 1, 2, 3, 4 objects respectively $= \dfrac{\lfloor 10}{\lfloor 1 \lfloor 2 \lfloor 3 \lfloor 4}$.

Number of ways to divde 10 non-identical objects in 4 groups (G_1, G_2, G_3, G_4) such that groups get objects in number 1, 2, 3, 4 (i.e. any group can get 1 object or 2 objects or 3 objects or 4 objects).

= Number of ways to divide and distribute 10 objects in 4 groups containing 1, 2, 3, 4 objects

$$= \frac{\lfloor 10}{\lfloor 1 \lfloor 2 \lfloor 3 \lfloor 4} \times \lfloor 4$$

6.2 Case - II : Equal division and distribution of non-identical objects

Here, we will see formulae to divide and distribute non-identical objects equally in groups i.e. each group gets equal numbers of objects. For example, dividing 6 objects in 3 groups such that each group gets 2 objects.

(a) Number of ways to divide (mn) objects equally in m group (each group gets n objects)

$$= \frac{\underline{|mn}}{\left(\underline{|n}\right)^m} \frac{1}{\underline{|m}}$$

(b) Number of ways to divide and distribute (mn) objects equally in m groups (each group gets n objects)

$$= \frac{\underline{|mn}}{\left(\underline{|n}\right)^m}$$

6.3 Case - III : Equal as well as Unequal Division and Distribution of non identical objects

Here, we will see formulae to divide and distribute non-identical objects into groups such that not all groups contain equal or unequal number of objects i.e. some groups get equal and some get unequal number of objects. For example, division of 6 objects in 4 groups containing 1, 1, 2, 2 objects.

(a) Number of ways to divide ($m + 2n + 3p$) objects in 6 groups containing m, n, n, p, p, p objects is :

$$= \frac{\underline{|m+2n+3p}}{\underline{|m}\left(\underline{|n}\right)^2\left(\underline{|p}\right)^3} \times \frac{1}{\underline{|2}} \times \frac{1}{\underline{|3}}$$

> ***Note :*** We divide by $\underline{|2}$ because there are two groups containing n objects each (equal number of objects).
> We divide by $\underline{|3}$ because these are 3 groups containing p objects each (equal number of objects).

(b) Number of ways to divide and distribute ($m + 2n + 3p$) objects in 6 groups containing m, n, n, p, p, p objects

= [Number of ways to divide ($m + 2n + 3p$) objects in 6 groups] × (Number of groups) !

$$= \left[\frac{\underline{|m+2n+3p}}{\underline{|m}\left(\underline{|n}\right)^2\left(\underline{|p}\right)^3} \frac{1}{\underline{|2}} \times \frac{1}{\underline{|3}}\right] \times \underline{|6}$$

(c) Number of ways to divide and distribute ($m + 2n + 3p$) objects in 6 groups containing m, n, n, p, p, p objects such that ($m + 2n + 3p$) objects are distributed among equal groups only

$$= \frac{\underline{|m+2n+3p}}{\underline{|m}\left(\underline{|n}\right)^2\left(\underline{|p}\right)^3}$$

Above formulae is defined for dividing objects in 6 groups but we can make similar formulae for other cases.

For example :

Number of ways to divide 10 objects in 4 groups containing 3, 3, 2, 2 objects

$$= \frac{\lfloor 10}{(\lfloor 2)^2 (\lfloor 3)^2} \frac{1}{\lfloor 2} \frac{1}{\lfloor 2}$$

Number of ways to divide and distribute completely 10 objects in 4 groups containing 3, 3, 2, 2 objects

$$= \left[\frac{\lfloor 10}{(\lfloor 2)^2} \frac{1}{(\lfloor 3)^2} \frac{1}{\lfloor 2} \frac{1}{\lfloor 2}\right] \times \lfloor 4$$

Number of ways to divide and 'distribute among equal groups' 10 objects containing 2, 2, 3, 3 objects

$$= \frac{\lfloor 10}{\lfloor 2 \lfloor 2 \lfloor 3 \lfloor 3}$$

6.4 Other Formulae :

(a) Number of ways to divide n non-identical objects in r groups such that each group gets 0 or more number of objects (empty groups are allowed) $= r^n$

(b) Number of ways to divide n non-identical objects in r groups such that each group gets at least one object (empty groups are not allowed)

$$= r^n - {}^rC_1 (r-1)^n + {}^rC_2 (r-2)^n - {}^rC_3 (r-3)^n + \ldots\ldots (-1)^{r-1}\ {}^rC_{r-1} 1^n$$

Illustrating the Concepts :

➤ *In how many ways can 12 books be equally distributed among 3 students ?*

In this question, we have to divide books equally among 3 students. So we will use formulae given in section 6.2. where we divide non - identical objects equally among groups.

Therefore, number of ways to divide and distribute 12 non-identical objects among 3 students equally $= \frac{\lfloor 12}{(\lfloor 4)^3}$

➤ *In how many ways we can divide 52 playing cards*

(i) *among 4 players equally ?* **(ii)** *in 4 equal parts ?*

(i) 52 cards are to be divided equally among 4 players. Each player will get 13 cards. If we distribute a way of division among players, it a different division. It means we should apply distribution formula. Using formula given in section 6.2 (b), we get :

Number of ways to divide playing cards $= \frac{\lfloor 52}{(\lfloor 13)^4}$

(ii) As we have to make 4 equals parts, each part consists of 13 cards. We will apply division formula (not distribution) because if we distribute a way of division among various parts, then it will be the same division Using formula used in section 6.2 (a), we get :

Number of ways to divide 52 cards in 4 parts $= \frac{\lfloor 52}{(\lfloor 13)^4} \frac{1}{\lfloor 4}$

Illustration - 46 *In how many ways can 7 departments be divided among 3 ministers such that every minister gets at least one and atmost 4 departments to control ?*

(A) 630 **(B)** 1050 **(C)** 1680 **(D)** *None of these*

SOLUTION : (C)

Let 3 ministers be M_1, M_2, M_3.

Following are the ways in which we can divide 7 departments among 3 ministers such that each minister gets at least one and atmost 4.

S.No.	M_1	M_2	M_3
1	4	2	1
2	2	2	3
3	3	3	1

Note : If we have a case (2, 2, 3), then there is no need to make cases (3, 2, 2) or (2, 3, 2) because we will include them when we apply distribution formula to distribute ways of division among ministers.

Case - I :

We divide 7 departments among 3 ministers in number 4, 2, 1 *i.e.* unequal division. As any minister can get 4 departments, any can get 2, any can get 1 department, we should apply distribution formula. Using formula given in section 6.1 (b), we get :

Number of ways to divide and distribute departments in number 4, 2, 1

$$= \left[\frac{\underline{|7}}{\underline{|4}\,\underline{|2}\,\underline{|1}}\right] \times \underline{|3} = 630 \qquad \textbf{. . . (i)}$$

Case - II :

It is 'equal as well as unequal' division. As any minister can get any number of departments, we use complete distribution formula. Using formula given in section 6.3 (b), we get :

Number of ways to divide and distribute departments in number 2, 2, 3,

$$= \left[\frac{\underline{|7}}{\underline{|2}\,\underline{|2}\,\underline{|3}}\;\frac{1}{\underline{|2}}\right] \times \underline{|3} = 630 \qquad \textbf{. . . (ii)}$$

Case - III :

It is also 'equal as well as unequal' division. As any minister can get any number of departments, we use complete distribution formula. Using formula given in section 6.3 (b), we get :

Number of ways to divide and distribute departments in number 3, 3, 1

$$= \left[\frac{\underline{|7}}{(\underline{|3})^2\,(\underline{|1})}\;\frac{1}{\underline{|2}}\right] \times \underline{|3} = 420 \qquad \textbf{. . . (iii)}$$

Combining (i), (ii) and (iii), we get number of ways to divide 7 departments among 3 ministers

$= 630 + 630 + 420 = 1680$